Sigmoid Approximation to the Gaussian Q-function and its Applications to Spectrum Sensing Analysis

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***Abstract*—Most of the existing approximations for the Gaussian Q-function have been developed bearing in mind applications that require high estimation accuracy for large argument values (e.g., derivation of the bit/symbol error rates of digital communication systems, which are typically in the order of 10^{-6} to 10^{-12}). Such values correspond to positive arguments of the function and consequently most of the existing approximations are valid for positive arguments only. However, other relevant problems where the Gaussian Q-function can appear do not require such a level of accuracy (e.g., derivation of the detection probability of a signal detector, where accuracies of two or three decimal figures are sufficient) and, more importantly, require the evaluation of the Q-function over the whole range of values (i.e., both positive and negative arguments). In this context, this paper analyses a sigmoid approximation to the Q-function that provides adequate levels of accuracy for any real argument. As an illustrative example, this approximation is employed to obtain new closed-form expressions for the probability of detection of an energy detector under Rayleigh and Nakagami-m fading channels.**

I. Introduction

The Gaussian Q-function $\mathcal{Q}(x)$ [1, eq. (26.2.3)] is found in many problems of digital communication systems. Since no exact and simple closed-form expression (adequate for mathematical manipulations) is known, several approximations have been proposed [2]–[11]. The existing approximations have been developed mainly for the derivation of the bit or symbol error rates of digital communication systems over fading channels, which are in the order of 10^{-6} to 10^{-12} [12] and therefore require accuracies of 6 to 12 decimal figures. This range of values corresponds to positive arguments of the Q-function ($x \geq 0$) and most of the existing approximations, despite their high accuracies within such region of arguments, are only valid for positive arguments, resulting in high estimation errors for negative arguments – and in some cases, for positive arguments close to zero as well. Only the polynomial approximation proposed in [6] is valid over a limited range of both positive and negative arguments around the origin. However, such approximation was specifically envisaged for analytical derivations of error rates in log-normal channels and its complex form is in general unsuitable for other scenarios.

Some relevant scenarios where the Gaussian Q-function can also appear do not require such high levels of accuracy and, more importantly, require the evaluation of the Q-function over the whole range of values (i.e., both positive and negative arguments). A good example of this is the derivation of the detection probability of signal detection methods (referred to as *spectrum sensing* methods in the context of cognitive radio [13]) over fading channels, where accuracies of two or three significant figures are sufficient for most practical applications. The integrals found in this type of problems usually require integrating $\mathcal{Q}(x)$ over positive and negative arguments, something for which the existing approximations are not well suited. A possible solution is to make use of the property $\mathcal{Q}(-x) = 1 - \mathcal{Q}(x)$, which enables the application of existing approximations to negative arguments. However, when integrating $\mathcal{Q}(x)$ over fading channels, this approach requires the original integral to be split into several integrals (of a different type in general), thus leading to tedious analytical developments (e.g., see the example in [8]). In this type of problems it would be desirable and convenient to have an approximation that provides the required level of accuracy (i.e., up to the second or third decimal figure) over the whole range of both positive and negative arguments. In this context, this paper evaluates an approximation to the Q-function that provides a satisfactory level of accuracy for any real argument.

II. Sigmoid Approximation to the Q-function

The Gaussian Q-function is defined as [1, eq. (26.2.3)]:

$$\mathcal{Q}(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} dt \tag{1}$$

The shape of the Gaussian Q-function resembles that of the sigmoid function (standard logistic function), which is defined as $1/(1+e^{-x})$, but with inverted symmetry around the origin. Motivated by this observation, the following approximation, based on a modified sigmoid function, is here considered:

$$\begin{aligned}\mathcal{Q}(x) \approx \widehat{\mathcal{Q}}(x) &= \frac{1}{1+e^{\alpha x}} = \frac{e^{-\alpha x}}{1+e^{-\alpha x}} = \frac{e^{-\frac{\alpha x}{2}}}{e^{\frac{\alpha x}{2}} + e^{-\frac{\alpha x}{2}}} \\ &= \frac{1}{2}\left[1 - \tanh\left(\frac{\alpha x}{2}\right)\right], \quad x \in \mathbb{R} \end{aligned} \tag{2}$$

where $\alpha \in \mathbb{R}$ is a fitting coefficient. Notice that all the forms in (2) are mathematically equivalent, but some may be more convenient depending on the particular integral to be solved.

The fitting coefficient α can be computed so as to minimise the root mean square error (RMSE) of the sigmoid approximation within the range of arguments of interest, $x \in [-\Upsilon, \Upsilon]$:

$$\alpha = \arg\min_\beta \sqrt{\frac{1}{2\Upsilon} \int_{-\Upsilon}^{\Upsilon} \left[\mathcal{Q}(x) - \frac{1}{1+e^{\beta x}}\right]^2 dx} \tag{3}$$

TABLE I

OPTIMUM VALUE OF α FOR DIFFERENT ARGUMENT RANGES.

Argument range	**Optimum α**
$x \in [-1, +1]$	1.6331
$x \in [-2, +2]$	1.6855
$x \in [-3, +3]$	1.6997
$x \in [-4, +4]$	1.7009
$x \in [-5, +5]$	1.7010
$x \in (-\infty, +\infty)$	1.7010

The optimum value of α depends on the considered argument range (Table I). As the argument range increases, the optimum value of α converges to a value. For argument ranges $\Upsilon \geq 5$, the result of evaluating (3) converges to $\alpha = 1.7010 \approx 1.7$, which constitutes the optimum value of α that minimises the RMSE of the sigmoid approximation for any real argument.

III. ACCURACY ANALYSIS

The relative error is usually employed to compare approximations to numbers of widely differing size (i.e., with differences of several orders of magnitude). As mentioned in Section I, most existing approximations have been proposed to derive the bit/symbol error rates of digital communication systems over fading channels, which can be in the order of 10^{-6} to 10^{-12}. Given this range of values of $\mathcal{Q}(x)$, the use of the relative error, which provides a finer detail of appreciation at such low values, comes as a natural choice for evaluating the accuracy of previous approximations. However, the relative error does not constitute an adequate metric of accuracy in this work. The reason is that the values of $\mathcal{Q}(x)$ for negative arguments ($\lim_{x\to-\infty} \mathcal{Q}(x) = 1$) are greater than the values for positive arguments ($\lim_{x\to\infty} \mathcal{Q}(x) = 0$). As a result, and despite the symmetry around the origin of $\mathcal{Q}(x)$ and the approximation in (2), the relative error (i.e., the absolute error divided by the true function's value) would incorrectly suggest that the considered approximation is more accurate for negative arguments. To avoid this artefact, the absolute error is used in this paper as accuracy metric instead of the relative error (notice that previous approximations are only valid for positive arguments and therefore do not suffer from this problem). Moreover, for the scenario considered in this paper (i.e., derivation of the detection probability of spectrum sensing methods over fading channels), accuracies of two or three significant figures are sufficient. In this case, the values of interest of $\mathcal{Q}(x)$ are of a more similar order of magnitude (i.e., from 10^0 to 10^{-2} or 10^{-3}) and therefore the absolute error constitutes an adequate metric of accuracy.

Fig. 1 compares the absolute error of the sigmoid and other existing approximations [2]–[10]. The results are shown with the ordinates axes in linear (top) and logarithmic (bottom) scales for different details of appreciation. As discussed in Section I, most of the existing approximations are valid for positive arguments and, as expected, are characterised by high estimation errors when evaluated at negative arguments. For these approximations, the property $\mathcal{Q}(-x) = 1 - \mathcal{Q}(x)$ can be used to evaluate the Q-function for negative arguments. With this strategy, the absolute errors for negative arguments would be a symmetric reproduction of the absolute errors shown in Fig. 1 for positive arguments. There are a few cases (Chiani, Loskot and Olabiyi) for which this strategy would still lead to relatively high estimation errors as a result of the inaccuracy of these approximations for positive arguments close to zero. For other approximations, this strategy would enable an accurate evaluation of the Q-function over the whole range of arguments at the expense of an increased complexity of analytical manipulations as discussed in Section I. As it can be appreciated, only the approximations proposed by Chen and in (2) are able to provide an absolute error of less than 10^{-2} (which is sufficient for the problem considered in this paper) over the whole range of arguments without resorting to strategies that lead to an increased analytical complexity. However, as mentioned in Section I, the approximation proposed by Chen was specifically envisaged for analytical derivations of error rates in log-normal channels and its complex form (see [6, eq. (4)]) is unsuitable for the scenario considered in this paper. Therefore, only the approximation in (2) provides an adequate level of accuracy for any real (positive or negative) argument of the Gaussian Q-function with a reasonable level of analytical complexity.

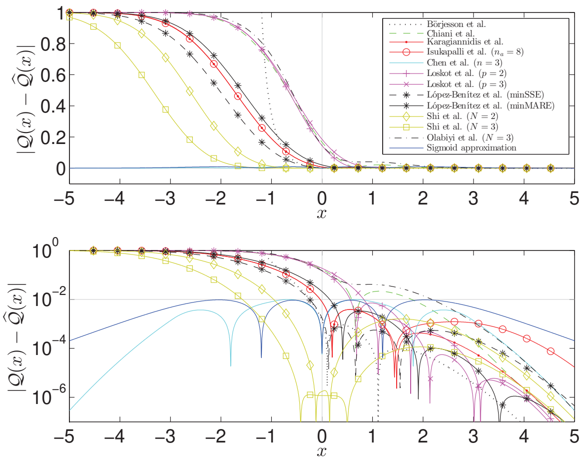

Fig. 1. Absolute error of the sigmoid and other approximations [2]–[10].

In summary, while some of the existing approximations can provide a better accuracy than the approximation in (2), they require the use of the property $\mathcal{Q}(-x) = 1 - \mathcal{Q}(x)$ for negative arguments of the Q-function (thus leading to an increased complexity in the resulting algebraic manipulations) and/or are characterised by expressions that are in general too complex to be employed in scenarios other than those for which they were conceived. On the other hand, the approximation in (2) provides the required level of accuracy at an affordable level of analytical complexity as it will be illustrated below.

IV. Applications

In contrast to previous studies where the purpose of approximating the Gaussian Q-function was the derivation of bit/symbol error rates over fading channels, this section uses the sigmoid approximation in (2) to compute the probability of detection of an energy detector over fading channels. The main difference between both problems is that a much lower level of accuracy is sufficient in the latter case but evaluation over both positive and negative arguments is required.

The probability of detection of an energy detector in an additive white Gaussian noise (AWGN) channel can be expressed as $P_d(\gamma) = \mathcal{Q}(\zeta(\gamma))$, with $\zeta(\gamma)$ given by [14]:

$$\zeta(\gamma) = \frac{\mathcal{Q}^{-1}(P_{fa})\sqrt{2N} - N\gamma}{\sqrt{2N}(1+\gamma)} \approx \mathcal{Q}^{-1}(P_{fa}) - \sqrt{\frac{N}{2}}\gamma \quad (4)$$

where N is the number of signal samples collected during the sensing interval, P_{fa} is the target probability of false alarm and γ is the instantaneous signal-to-noise ratio (SNR) per symbol of the channel. The approximation in the right-hand side of (4) assumes the common case of low SNR regime (i.e., $\gamma \ll 1$).

$P_d(\gamma)$ gives the probability of detection conditioned on the instantaneous SNR, γ. Under varying SNR, a more useful performance parameter is the average probability of detection $\overline{P}_d$ experienced for an average SNR $\overline{\gamma}$, which can be obtained by averaging $P_d(\gamma)$ over the SNR statistics [12]:

$$\begin{aligned}\overline{P}_d(\overline{\gamma}) = \mathbb{E}\{P_d(\gamma)\} &= \int_0^\infty P_d(\gamma) f_\gamma(\gamma) d\gamma && (5)\\ &= \int_0^\infty \mathcal{Q}(\zeta(\gamma)) f_\gamma(\gamma) d\gamma && (6)\\ &\approx \int_0^\infty \frac{1}{1+e^{\alpha\zeta(\gamma)}} f_\gamma(\gamma) d\gamma && (7)\end{aligned}$$

where $f_\gamma(\gamma)$ is the probability density function (PDF) of the received SNR. Notice that the argument of $\mathcal{Q}(x)$ in (6), $\zeta(\gamma)$, can take both positive and negative values even though $\gamma \geq 0$. Some of the existing approximations could be employed to solve (6) by splitting the integral into one integral for $\zeta(\gamma) \geq 0$, where $\mathcal{Q}(\zeta(\gamma))$ is used, and another integral for $\zeta(\gamma) < 0$, where $1 - \mathcal{Q}(-\zeta(\gamma))$ is used, which in turn leads to two integrals. The resulting three integrals have different algebraic forms and require individual resolutions (see the example in [8]). On other hand, the approximation in (2), which is valid for positive and negative arguments, can be introduced into (6) without further rearrangements, leading to a single integral as shown in (7). As appreciated, the sigmoid approximation greatly simplifies the resolution of the integral.

The following subsections illustrate the resolution of the integral in (7) for several cases of practical interest.

A. Rayleigh fading

Under Rayleigh fading, the instantaneous SNR per symbol follows an exponential distribution given by [12, eq. (2.7)]:

$$f_\gamma(\gamma) = \frac{1}{\overline{\gamma}} \exp\left(-\frac{\gamma}{\overline{\gamma}}\right), \quad \gamma \geq 0 \quad (8)$$

where $\overline{\gamma}$ is the average SNR per symbol.

Introducing (8) into (7) leads to the following integral:

$$\overline{P}_d(\overline{\gamma}) \approx \int_0^\infty \frac{\frac{1}{\overline{\gamma}} e^{-\frac{\gamma}{\overline{\gamma}}}}{1 + e^{\alpha\left[\mathcal{Q}^{-1}(P_{fa}) - \sqrt{\frac{N}{2}}\gamma\right]}} d\gamma \quad (9)$$

Applying the change of variable $\upsilon = e^{-\alpha\sqrt{\frac{N}{2}}\gamma}$, the resulting integral on υ can be solved with the aid of [15, eq. (3.194.5)]:

$$\overline{P}_d(\overline{\gamma}) \approx {}_2F_1\left(1, \frac{1}{\alpha\sqrt{\frac{N}{2}}\overline{\gamma}}; 1 + \frac{1}{\alpha\sqrt{\frac{N}{2}}\overline{\gamma}}; -e^{\alpha\mathcal{Q}^{-1}(P_{fa})}\right) \quad (10)$$

where ${}_2F_1(\cdot)$ represents the Gauss hypergeometric function, whose definition can be found in [15, eqs. (9.14) & (9.111)].

The performance of energy detection under Rayleigh fading has also been studied in [8], applying the property $\mathcal{Q}(\zeta(\gamma)) = 1 - \mathcal{Q}(-\zeta(\gamma))$ for $\zeta(\gamma) < 0$, and in [16], obtaining in both cases mathematical expressions of notable complexity (see [8, eq. (21)] and [16, eq. (9)]). However, the sigmoid approximation yields the much simpler result shown in (10).

B. Nakagami-m fading

Under Nakagami-m fading, the instantaneous SNR per symbol follows a gamma distribution given by [12, eq. (2.21)]:

$$f_\gamma(\gamma) = \frac{m^m \gamma^{m-1}}{\overline{\gamma}^m \Gamma(m)} \exp\left(-\frac{m\gamma}{\overline{\gamma}}\right), \quad \gamma \geq 0 \quad (11)$$

where $m \geq 1/2$ is the Nakagami-m fading parameter and $\Gamma(\cdot)$ is the gamma function [1, 6.1.1].

Introducing (11) into (7) leads to the integral in (13), where:

$$\mathcal{I}(\gamma) = \int \frac{\gamma^{m-1} e^{-\frac{m\gamma}{\overline{\gamma}}}}{1 + e^{\alpha\left[\mathcal{Q}^{-1}(P_{fa}) - \sqrt{\frac{N}{2}}\gamma\right]}} d\gamma \quad (12)$$

The integral in (12) can be solved for individual integer values of m ($m = 1, 2, \ldots$) and, based on the obtained solutions, the pattern for any integer m can be inferred, leading to the result in (14) where $\operatorname{sgn}(x) = \frac{x}{|x|}$ is the sign function and ${}_{k+1}F_k(\cdot)$ is the generalised hypergeometric function [15, eq. (9.14.1)].

When $\gamma \to 0$, all the terms of the sum in (14) are zero except for $k = m$, hence (15). When $\gamma \to \infty$, the limit of the terms of the sum in (14) alternates between $\pm\infty$ for even/odd values of k, which hinders the calculation of $\lim_{\gamma\to\infty} \mathcal{I}(\gamma)$. However, it can be verified that there exists an SNR value $\gamma = \xi$ above which the value of the integral in (14) remains constant such that $\lim_{\gamma\to\infty} \mathcal{I}(\gamma) \approx \mathcal{I}(\xi)$. The value of ξ can be obtained based on the SNR distribution. Notice that the integral in (12) is implicitly associated with the integral of the SNR PDF and therefore its result in (14) can be associated with the corresponding cumulative distribution function (CDF) of the SNR. If the SNR CDF is set equal to a sufficiently high percentile ρ (e.g., $\rho = 0.9999$), then the corresponding SNR value $\gamma = \xi$ guarantees that the CDF remains nearly constant (equal to one) for any SNR greater than ξ and so does the

$$\overline{P}_d(\overline{\gamma}) \approx \frac{m^m}{\overline{\gamma}^m \Gamma(m)} \int_0^\infty \frac{\gamma^{m-1} e^{-\frac{m\gamma}{\overline{\gamma}}}}{1+e^{\alpha\left[\mathcal{Q}^{-1}(P_{fa})-\sqrt{\frac{N}{2}}\gamma\right]}} d\gamma = \frac{m^m}{\overline{\gamma}^m \Gamma(m)} \left[\mathcal{I}(\gamma)\right]_0^\infty = \frac{m^m}{\overline{\gamma}^m \Gamma(m)} \left[\lim_{\gamma\to\infty} \mathcal{I}(\gamma) - \lim_{\gamma\to 0} \mathcal{I}(\gamma)\right] \tag{13}$$

$$\mathcal{I}(\gamma) = -\left(e^{-\frac{m\gamma}{\overline{\gamma}} - \alpha\zeta(\gamma)}\right) \sum_{k=1}^{m} \left(\prod_{j=0}^{\max(0,k-2)} (m-j-1)^{\frac{\operatorname{sgn}\left(k-\frac{3}{2}\right)+1}{2}}\right) \frac{\gamma^{m-k}}{\left(\frac{m}{\overline{\gamma}} - \alpha\sqrt{\frac{N}{2}}\right)^k} \cdot$$
$$\cdot\, {}_{k+1}F_k\left(1, \underbrace{1-\frac{m\sqrt{\frac{2}{N}}}{\alpha\overline{\gamma}}, \ldots, 1-\frac{m\sqrt{\frac{2}{N}}}{\alpha\overline{\gamma}}}_{k \text{ times}};\ \underbrace{2-\frac{m\sqrt{\frac{2}{N}}}{\alpha\overline{\gamma}}, \ldots, 2-\frac{m\sqrt{\frac{2}{N}}}{\alpha\overline{\gamma}}}_{k \text{ times}};\ -e^{-\alpha\zeta(\gamma)}\right) \tag{14}$$

$$\mathcal{I}(0) = \lim_{\gamma\to 0} \mathcal{I}(\gamma) = -\frac{e^{-\alpha\mathcal{Q}^{-1}(P_{fa})}}{\left(\frac{m}{\overline{\gamma}} - \alpha\sqrt{\frac{N}{2}}\right)^m} \left(\prod_{j=0}^{\max(0,m-2)} (m-j-1)^{\frac{\operatorname{sgn}\left(m-\frac{3}{2}\right)+1}{2}}\right) \cdot$$
$$\cdot\, {}_{m+1}F_m\left(1, \underbrace{1-\frac{m\sqrt{\frac{2}{N}}}{\alpha\overline{\gamma}}, \ldots, 1-\frac{m\sqrt{\frac{2}{N}}}{\alpha\overline{\gamma}}}_{m \text{ times}};\ \underbrace{2-\frac{m\sqrt{\frac{2}{N}}}{\alpha\overline{\gamma}}, \ldots, 2-\frac{m\sqrt{\frac{2}{N}}}{\alpha\overline{\gamma}}}_{m \text{ times}};\ -e^{-\alpha\mathcal{Q}^{-1}(P_{fa})}\right) \tag{15}$$

result in (14). The CDF of the instantaneous SNR per symbol under Nakagami-m fading is given by:

$$F_\gamma(\gamma) = P\left(m, \frac{m\gamma}{\overline{\gamma}}\right) = \frac{1}{\Gamma(m)} \int_0^{\frac{m\gamma}{\overline{\gamma}}} e^{-t} t^{m-1} dt, \quad \gamma \geq 0 \tag{16}$$

where $P(\cdot)$ is the regularised lower incomplete gamma function [1, eq. (6.5.1)]. Setting $F_\gamma(\gamma) = \rho$ and solving for γ, the corresponding value of ξ is obtained as:

$$\xi = \frac{\overline{\gamma}}{m} P^{-1}(m, \rho) \tag{17}$$

where $P^{-1}(\cdot)$ is the inverse of $P(\cdot)$.

Finally, the introduction of the value of the limits in (13) yields the detection probability under Nakagami-m fading:

$$\overline{P}_d(\overline{\gamma}) \approx \frac{m^m}{\overline{\gamma}^m \Gamma(m)} \left[\mathcal{I}(\xi) - \mathcal{I}(0)\right] \tag{18}$$

C. Numerical Results and Discussion

Fig. 2 (Rayleigh), Fig. 3 (Nakagami, $m = 2$), Fig. 4 (Nakagami, $m = 3$) and Fig. 5 (Nakagami, $m = 4$) compare the results in (10) and (18), obtained based on the approximation in (2), with their exact counterparts, obtained by integrating (6) numerically with the corresponding PDFs shown in (8) and (11). In each figure, the graph on the top shows the average probability of detection $\overline{P}_d(\overline{\gamma})$ as a function of the average SNR (only for $P_{fa} = 0.01$ for the sake of clarity), while the graph on the bottom shows the absolute error with respect to the exact values (for $P_{fa} = 0.01$ and $P_{fa} = 0.10$). The graphs show groups of three curves, which correspond to $N = 10^2$ (right), $N = 10^3$ (middle) and $N = 10^4$ (left).

As appreciated, the results obtained with (10) and (18) are highly accurate, with approximation errors of less than 1% in all cases (for $P_{fa} = 0.01$) or even less (for $P_{fa} = 0.10$). This level of accuracy is more than enough for most practical applications. It is interesting to highlight that for the typical operation point of spectrum sensing algorithms (i.e., high values of detection probability), the results in (10) and (18) are nearly exact. These results demonstrate the applicability and benefits of the sigmoid approximation to the Q-function.

V. Conclusions

This paper has evaluated a sigmoid approximation to the Gaussian Q-function. As opposed to most existing approximations, which have been designed to provide high accuracy for large (positive only) arguments of the function, the sigmoid approximation has been proven to be a more suitable alternative for applications that do not require such high levels of accuracy (e.g., up to the second or third decimal figure) but need to be evaluated over both positive and negative arguments (something for which most existing approximations are not suitable as they result in significantly more complex algebraic manipulations). The applicability of the sigmoid approximation has been illustrated in the context of performance analysis of spectrum sensing in cognitive radio over fading channels. Nevertheless, it may also find applications in other areas with similar requirements as those of the example here considered.

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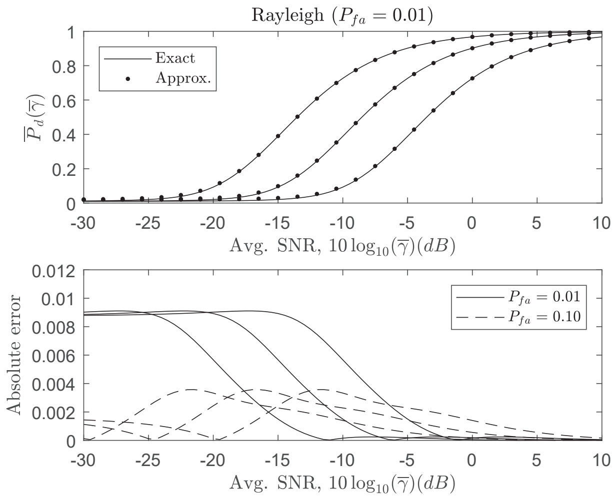

Fig. 2. Approximated and exact detection probabilities (Rayleigh).

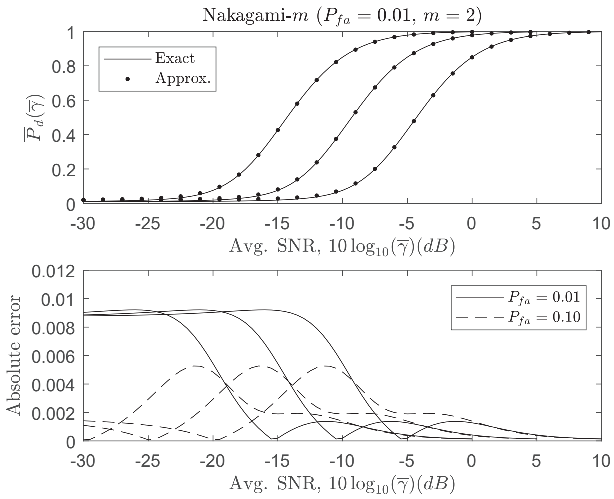

Fig. 3. Approximated and exact detection probabilities (Nakagami, $m = 2$).

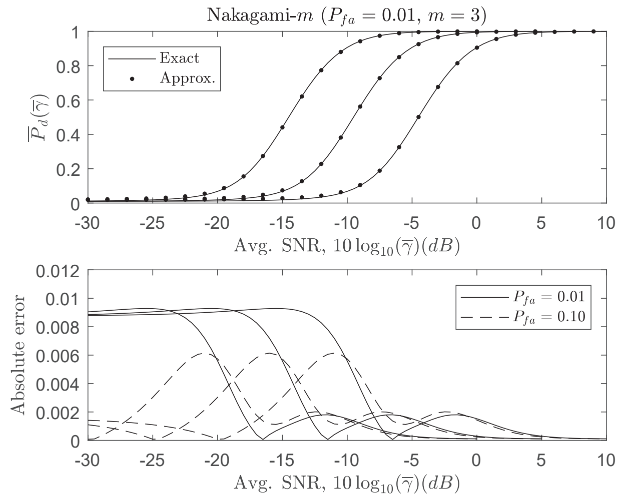

Fig. 4. Approximated and exact detection probabilities (Nakagami, $m = 3$).

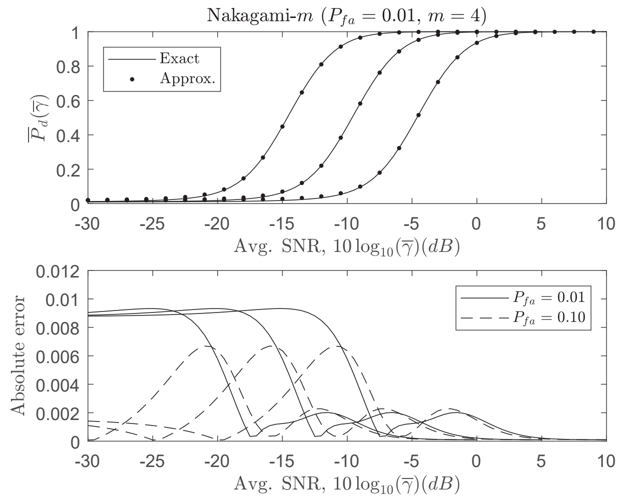

Fig. 5. Approximated and exact detection probabilities (Nakagami, $m = 4$).

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